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Dynamically twisted algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ as current algebra generalizing screening currents of q-deformed Virasoro algebra

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Abstract. In this paper, we propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ which is based on the relations $RLL = LLR^*$, where R and R^* are the dynamical R-matrices of $A_1^{(1)}$-type face model with the elliptic moduli shifted by the centre of the algebra. From the Ding–Frenkel correspondence, we find that its corresponding (Drinfeld) current algebra at level one is the algebra of screening currents for q-deformed Virasoro algebra. We realize the elliptic algebra at level one by Miki's construction from the bosonization for the type I and type II vertex operators. We also show that the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is related to the algebra $A_{q,p}(\widehat{gl_2})$ by dynamically twisting.

1. Introduction

As the quantum form of fundamental Poisson bracket, the '$RLL = LLR$' relations (or 'RLL' formalism) define various quantum algebra which appear in quantum field theory (QFT) and statistical mechanics. It is associated with structure constants—the R-matrix satisfying the Yang–Baxter equation [1, 2]. Drinfeld and Jimbo have discovered a fundamental algebra structure—quantum algebra $U_q(g)$ where g is some finite- or infinite-dimensional Lie algebra [3, 4]. Faddeev, Reshetikhin and Takhtajan [5] realize the algebra $U_q(g)$ (FRT construction), where g is some finite-dimensional Lie algebra, by the 'RLL' formalism with a spectral parameter independent R-matrix. Later, Reshetikhin and Semenov-Tian-Shansky [6] constructed a new realization of q-deformed affine algebra by the 'RLL' formalism with a trigonometric R-matrix (which are the so-called RS relations) characterized by the spectral parameter shifted with the centre of the algebra. Ding and Frenkel [7] gave the isomorphism between the realization given by Reshetikhin and Semenov-Tian-Shansky and the Drinfeld realization of q-deformed affine algebra. Moreover, Khoroshkin [8] successfully constructed the realization of Yangian Double with centre $DY_{\hbar}(\hat{g})$ in the'RLL' formalism [9], which is associated with the rational R-matrix. Foda *et al* proposed an elliptic extension of the quantum affine algebra $A_{q,p}(\widehat{sl_2})$ [10] as a symmetric algebra for the eight-vertex model. The elliptic algebra is based on the generalized 'RLL' formalism: $RLL = LLR^*$, where R and R^* are eight-vertex R-matrices with elliptic moduli differing by an amount depending on the level k of the representation

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on which L acts. The algebra $A_{\hbar,\eta}(\widehat{sl_2})$ as the scaling limit of the elliptic algebra $A_{q,p}(\widehat{sl_2})$ was formulated in the '$RLL = LLR^*$' formulation by Jimbo *et al* [36] and was studied by Khoroshikin *et al* [8] through the Gauss decomposition. In fact, the above-mentioned progress in 'RLL' formalism all involve vertex-type models.

Another progress of quantum algebra focuses on the q-deformation [11–13] and $\hbar$-deformation [14] of Virasoro and W algebra, and q-deformed extended Virasoro algebra (suggested by Konno in [27]), which would play almost the same role in off-critical integrable model as that of Virasoro, W algebra and extended Virasoro algebra [38–40] in two-dimensional conformal field theories (CFT) [15] for the critical model. q-deformed Virasoro (q-Virasoro) algebra and q-deformed extended Virasoro (extended q-Virasoro) algebra arise in the two-dimensional solvable lattice models [12, 17, 27] (e.g. ABF model [19] etc); $\hbar$-deformed Virasoro ($\hbar$-Virasoro) algebra is studied as the hidden symmetry of the massive integrable field models (e.g. the restricted sine-Gordon model [14]). It was shown that the screening currents for q-Virasoro algebra [12, 13, 20, 21, 41] and $\hbar$-Virasoro algebra [14] satisfy a closed algebra relation which is some further deformation of q-affine algebra and Yangian double with centre. In another way, q-Virasoro algebra, extended q-Virasoro and $\hbar$-Virasoro algebra can be reconstructed as the algebra which commutes with the screening currents up to a total difference [17, 42, 43]. Apparently, they constitute the hidden symmetries in the $A_1^{(1)}$-type face model [12]. So, the studies of the algebraic structure of the screening currents for q-Virasoro, extended q-Virasoro and $\hbar$-Virasoro are of great importance. In this paper, we deal mainly with the screening algebra for q-Virasoro algebra and possibly the extended q-Virasoro, as a byproduct, the screening algebra for $\hbar$-Virasoro algebra and related $\hbar$-deformed extended Virasoro algebra can be obtained by taking the scaling limit of the q-deformed version.

The 'RLL' formalism was originally formulated for the nondynamical Yang–Baxter equation (or the vertex-type models), Felder [25] succeeded in extending it to incorporate the dynamical Yang–Baxter equations (Gervais–Neveu–Felder equation) [22] which is associated with the q-deformation of the Knizhnik-Zamolodchikov-Bernard equation on a torus. In fact, the 'RLL' formalism given by Felder [25] and Enriquez *et al* [22] is a dynamical version of the FRT construction and RS relations respectively. In this paper, we extend the works of Fodal *et al* [10] to the dynamical R-matrix case. Namely, we propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ based on the relations $RLL = LLR^*$, where R and R^* are the dynamical R-matrices for $A_1^{(1)}$ face model (i.e. the solution to the star–triangle relation in the $A_1^{(1)}$-type face model) with elliptic moduli shifted by the centre of the algebra. Using the Ding–Frenkel correspondence, we construct Drinfeld currents for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. From the Drinfeld currents for the algebra (which is a subalgebra of $A_{q,p;\hat{\pi}}(\widehat{gl_2})$), we show that the (Drinfeld) current algebra at level one and a higher level is just the algebra of screening currents for q-Virasoro algebra and the algebra of screening currents for the extended q-Virasoro algebra [27] respectively. The algebra of screening currents at level one was studied by Awata *et al* [12] for q-Virasoro algebra and by Feigin *et al* [13, 21] for q-deformed W algebra, which is some elliptic deformation of affine algebra. The elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at a higher level would play an important role in the studies of the fusion ABF models [26, 29] and relate to the extended q-Virasoro algebra. Moreover, the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is the dynamical twisted algebra [23–25, 44] of the elliptic algebra $A_{q,p}(\widehat{gl_2})$.

This paper is organized as follows. Section 2, after reviewing q-Virasoro algebra, we introduce the algebra of screening currents for q-Virasoro algbera. In section 3, we construct an elliptic algbera $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ in terms of an $L^{\pm}$-operator which satisfies the dynamical relations of $RLL = LLR^*$ formulation. From Ding–Frenkel correspondence, we find that the corresponding dynamical Drinfeld current form a subalgebra which structure constants do not depend upon the dynamical variable and is just the algebra of screening currents for extended q-Virasoro algebra. The twisted relations between the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and $A_{q,p}(\widehat{gl_2})$ is constructed. In section 4, the bosonization of the type I and type II vertex operator for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one is constructed. By the Miki's construction, we obtained the bosonization for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one. The corresponding generalizing algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ for $\hbar$-deformed Virasoro algebra and $\hbar$-deformed extended Virasoro algebra, which is the scaling limit of the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$, is studied in section 5. Finally, we give a summary and discussions in section 6. The appendix contains some detailed calculations.

2. Algebra of screening currents for q-Virasoro algebra

We start by defining q-Virasoro algebra and the corresponding quantum Miura transformation.

2.1. q-Virasoro algebra and the quantum q-deformed Miura transformation

Let w be a generic complex number with $\mathrm{Im}(w) > 0$ and r be a real number with $4 < r$, and set $x = \mathrm{e}^{\mathrm{i}\pi w}$. Define elliptic functions

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{m\in Z} \exp\{\mathrm{i}\pi[(m+a)^2\tau + 2(m+a)(z+b)]\} \qquad \mathrm{Im}(\tau) > 0$$

$$\sigma_\alpha = \sigma_{(\alpha_1,\alpha_2)}(z,\tau) = \theta\begin{bmatrix} \frac{1}{2}+\frac{\alpha_1}{2} \\ \frac{1}{2}+\frac{\alpha_2}{2} \end{bmatrix}(z,\tau)$$

$$\theta^{(k)}(z,\tau) = \theta\begin{bmatrix} -\frac{k}{2} \\ 0 \end{bmatrix}(z,2\tau).$$

We shall use the following abbreviation

$$[v]_t = x^{\frac{v^2}{t}-v}\Theta_{x^{2t}}(x^{2v}) = \sigma_0\left(\frac{v}{t}, -\frac{1}{tw}\right) \times \text{constant} \qquad 0 < t$$

$$\Theta_q(z) = (z;q)(qz^{-1};q)(q;q) \qquad (z;q_1,\ldots,q_m) = \prod_{i_1,\ldots i_m=0}^{\infty}(1 - zq^{i_1}\ldots q^{i_m}).$$

The q-Virasoro algebra is generated by $\{T(z)\}$ with the following relations [11–13]

$$f\left(\frac{w}{z}\right)T(z)T(w) - f\left(\frac{z}{w}\right)T(w)T(z)$$
$$= \frac{(x^r - x^{-r})(x^{(r-1)} - x^{-(r-1)})}{x - x^{-1}}\left(\delta\left(\frac{w}{x^2z}\right) - \delta\left(\frac{x^2w}{z}\right)\right) \tag{1}$$

where $\delta(z) = \sum_{n\in Z} z^n$ and

$$f(z) = (1-z)^{-1}\frac{(zx^{2r};x^4)(zx^{2-2r};x^4)}{(zx^{2+2r};x^4)(zx^{4-2r};x^4)}.$$

The generators $T(z)$ for q-Virasoro algebra can be obtained by the following quantum q-deformed Miura transformation

$$T(z) = \Lambda(x^{-1}z) + \Lambda^{-1}(xz). \tag{2}$$

Define q-deformed bosonic oscillators $\beta_m (m \in Z/\{0\})$

$$[\beta_m, \beta_n] = m\frac{(x^m - x^{-m})(x^{(r-1)m} - x^{-(r-1)m})}{(x^{2m} - x^{-2m})(x^{rm} - x^{-rm})}\delta_{m+n,0} \tag{3}$$

and zero-mode operators P and Q such that $[P, \mathrm{i}Q] = 1$.

Then the fundamental operator $\Lambda(z)$ can be realized by q-deformed bosonic oscillators (see equation (3)) as follows

$$\Lambda(z) = x^{\sqrt{2r(r-1)}P} : \exp\left\{\sum_{m\neq}^{\infty}(x^{rm} - x^{-rm})\frac{\beta_m}{m}z^{-m}\right\}. \tag{4}$$

2.2. Algebra of screening currents

Let us introduce the screening currents $E(v)$, $F(v)$ for q-Virasoro algebra

$$E(v) = \mathrm{e}^{\mathrm{i}\sqrt{\frac{2(r-1)}{r}}(Q-\mathrm{i}2v\ln x P)} : \mathrm{e}^{\sum_{m\neq 0}\frac{x^m+x^{-m}}{m}\beta_m x^{-2vm}} : \tag{5}$$

$$F(v) = \mathrm{e}^{-\mathrm{i}\sqrt{\frac{2r}{r-1}}(Q-\mathrm{i}2v\ln x P)} : \mathrm{e}^{-\sum_{m\neq 0}\frac{x^m+x^{-m}}{m}\beta'_m x^{-2vm}} : \tag{6}$$

where $\beta'_m = \frac{x^{rm}-x^{-rm}}{x^{(r-1)m}-x^{-(r-1)m}}\beta_m$. Besides the well known bosonic realization of screening currents $E(v)$, $F(v)$, let us introduce $H^{\pm}$

$$H^-(v) = -x^{-4v} : E(v + \tfrac{1}{4})F(v - \tfrac{1}{4}) : \tag{7}$$

$$H^+(v) = -x^{-4v} : E(v - \tfrac{1}{4})F(v + \tfrac{1}{4}) : . \tag{8}$$

The screening currents commute with the generators of q-Virasoro algebra up to a total difference and form a closed algebra. In fact, from the normal order in appendix A, one can find that the screening currents defined in equations (5)–(8) realize an algebra satisfying the relations

$$E(v_1)E(v_2) = \frac{[v_1 - v_2 - 1]_r}{[v_1 - v_2 + 1]_r}E(v_2)E(v_1) \tag{9}$$

$$F(v_1)F(v_2) = \frac{[v_1 - v_2 + 1]_{r-1}}{[v_1 - v_2 - 1]_{r-1}}F(v_2)F(v_1) \tag{10}$$

$$[E(v_1), F(v_2)] = \frac{1}{x - x^{-1}}\{\delta(v_1 - v_2 + \tfrac{1}{2})H^+(v_1 + \tfrac{1}{4}) - \delta(v_1 - v_2 - \tfrac{1}{2})H^-(v_1 - \tfrac{1}{4})\} \tag{11}$$

$$H^{\pm}(v_1)E(v_2) = \frac{[v_1 - v_2 - 1 \mp \frac{1}{4}]_r}{[v_1 - v_2 + 1 \mp \frac{1}{4}]_r}E(v_2)H^{\pm}(v_1) \tag{12}$$

$$H^{\pm}(v_1)F(v_2) = \frac{[v_1 - v_2 + 1 \pm \frac{1}{4}]_{r-1}}{[v_1 - v_2 - 1 \pm \frac{1}{4}]_{r-1}}F(v_2)H^{\pm}(v_1) \tag{13}$$

$$H^{\pm}(v_1)H^{\pm}(v_2) = \frac{[v_1 - v_2 + 1]_{r-1}[v_1 - v_2 - 1]_r}{[v_1 - v_2 - 1]_{r-1}[v_1 - v_2 + 1]_r}H^{\pm}(v_2)H^{\pm}(v_1) \tag{14}$$

$$H^+(v_1)H^-(v_2) = \frac{[v_1 - v_2 + 1 + \frac{1}{2}]_{r-1}[v_1 - v_2 - 1 - \frac{1}{2}]_r}{[v_1 - v_2 - 1 + \frac{1}{2}]_{r-1}[v_1 - v_2 + 1 - \frac{1}{2}]_r}H^-(v_2)H^+(v_1) \tag{15}$$

where $H^-(v) = H^+(v + \frac{1}{2} - r)$. The algebra of screening currents written by Awata [12] has similar algebraic relations to ours. Actually, the screening currents defined in equations (9)–(15) realize the (Drinfeld) current algebra of an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one, which will be given by 'RLL' formalism in the following section.

3. The dynamical algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$

We propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ based on a dynamical $RLL = LLR^*$ relation. Then using Ding–Frenkel correspondence, we shall show that its Drinfeld current algebra is related to the current algebra generalizing the screening currents for q-Virasoro algebra. Moreover, algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is an algebraic structure underlying the elliptic solution to the star–triangle relation in $A_1^{(1)}$-face type model including ABF [19] and its fused version [26, 27, 29].

3.1. The R-matrix

Define a dynamical elliptic R-matrix (R-matrix for $A_1^{(1)}$ face model [28, 30])

$$R_F(v,\hat{\pi}) \equiv R_F(v,\hat{\pi},r) = \begin{pmatrix} a & & & \\ & b & c & \\ & d & e & \\ & & & a \end{pmatrix} \tag{16}$$

where $\hat{\pi}$ is the dynamical variable corresponding to the height for the face-type model and indulge in some relations with algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ (see equations (27) (28)). The matrix elements of the R-matrix is defined by

$$a(v,\hat{\pi}) = x^{\frac{1-r}{r}v}\frac{g_1(v)}{g_1(-v)} \qquad g_1(v) = \frac{\{x^{2+2v}\}\{x^{2+2r+2v}\}}{\{x^{4+2v}\}\{x^{2r+2v}\}} \qquad \{z\} = (z; x^{2r}, x^4) \tag{17}$$

$$\frac{b(v,\hat{\pi})}{a(v,\hat{\pi})} = \frac{[v]_r[\hat{\pi}-1]_r}{[v+1]_r[\hat{\pi}]_r} \qquad \frac{c(v,\hat{\pi})}{a(v,\hat{\pi})} = \frac{[v+\hat{\pi}]_r[1]_r}{[v+1]_r[\hat{\pi}]_r} \tag{18}$$

$$\frac{d(v,\hat{\pi})}{a(v,\hat{\pi})} = \frac{[\hat{\pi}-v]_r[1]_r}{[v+1]_r[\hat{\pi}]_r} \qquad \frac{e(v,\hat{\pi})}{a(v,\hat{\pi})} = \frac{[v]_r[1+\hat{\pi}]_r}{[v+1]_r[\hat{\pi}]_r}. \tag{19}$$

One can see that $a(v,\hat{\pi})$ does not depend on the dynamical variable $\hat{\pi}$. Moreover, let us introduce two R-matrices $R_F^{\pm}$ which coincide with R_F in equation (16) up to scalar factors independent of the dynamical variable

$$\begin{aligned} &R_F^{\pm}(v,\hat{\pi}) \equiv R_F^{\pm}(v,\hat{\pi},r) = \tau^{\pm}(v)R_F(v,\hat{\pi}) \qquad \tau^{\pm}(v) = \tau(-v \pm \tfrac{1}{2}) \\ &\tau(v) = x^{-v}\frac{(x^{1+2v};x^4)(x^{3-2v};x^4)}{(x^{3+2v};x^4)(x^{1-2v};x^4)} \end{aligned} \tag{20}$$

where $\tau^{\pm}(v)$ are the same as that of Foda *et al* [10]. $R_F^{\pm}$ are regarded as linear operators on $V \otimes V$, with $V = \mathrm{span}\{\mathrm{e}_{\pm}\}$. Let h be the diagonal 2×2 matrix $\mathrm{Diag}(1,-1)$ such that $he_{\pm} = \pm e_{\pm}$. The dynamical R-matrices $R_F^{\pm}(v,\hat{\pi})$ satisfy the dynamical Yang–Batxer equation (i.e. the modified Yang–Baxter equation [22]) in $V\otimes V\otimes V$

$$\begin{aligned} &R_{F12}^{\pm}(v_1-v_2,\hat{\pi}-2h^{(3)})R_{F13}^{\pm}(v_1-v_3,\hat{\pi})R_{F23}^{\pm}(v_2-v_3,\hat{\pi}-2h^{(1)}) \\ &\qquad = R_{F23}^{\pm}(v_2-v_3,\hat{\pi})R_{F13}^{\pm}(v_1-v_3,\hat{\pi}-2h^{(2)})R_{F12}^{\pm}(v_1-v_2,\hat{\pi}). \end{aligned} \tag{21}$$

Here we choose the same notation as Enriquez *et al* [22]: $R_{F12}^{\pm}(v,\hat{\pi}-2h^{(3)})$ means that if $a\otimes b\otimes e_\mu \in V\otimes V\otimes V$, $\mu \in \pm$, then $R_{F12}^{\pm}(v,\hat{\pi}-2h^{(3)})a\otimes b\otimes e_\mu = R_{F12}^{\pm}(v,\hat{\pi}-2\mu)a\otimes b\otimes e_\mu$, and the other symbols have a similar meaning. Besides the dynamical Yang–Baxter equation equation (21), the R-matrices have the following properties:

$$\text{unitarity } R_{F12}^{\pm}(v,\hat{\pi})R_{F21}^{\mp}(-v,\hat{\pi}) = \mathrm{id} \tag{22}$$

$$\text{crossing relations } R_F^{\pm}(-1-v,\hat{\pi})_{\mu v}^{\mu' v'} = \mu\mu' R_F^{\mp}(v,\hat{\pi}-\mu')_{-v'\mu}^{-v\mu'}\frac{[\hat{\pi}-\mu']_r}{[\hat{\pi}]_r}. \tag{23a}$$

Moreover, the R-matrices $R_F^{\pm}(v,\hat{\pi})$ have the following property

$$R_F^{+}(v+r,\hat{\pi}) = R_F^{-}(v,\hat{\pi}). \tag{23b}$$

Here, if $R_{F12}^{\pm}(v,\hat{\pi}) = \sum a_i \otimes b_i$, with $a_i, b_i \in \mathrm{End}(V)$, then $R_{F21}^{\pm}(v,\hat{\pi}) = \sum b_i \otimes a_i$.

3.2. The algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$

Let us proceed to the definition of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. Consider $L^{\pm}$-operators

$$L^{\pm}(v,\hat{\pi}) = \begin{pmatrix} L_{++}^{\pm} & L_{+-}^{\pm} \\ L_{-+}^{\pm} & L_{--}^{\pm} \end{pmatrix}$$

whose matrix elements are the generators of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ given by the following commutation relations:

$$R_F^{+}\left(v_1 - v_2 + \frac{c}{2},\hat{\pi}\right) L_1^{+}(v_1,\hat{\pi}) L_2^{-}(v_2,\hat{\pi}) = L_2^{-}(v_2,\hat{\pi}) L_1^{+}(v_1,\hat{\pi}) R_F^{*+}\left(v_1 - v_2 - \frac{c}{2},\hat{\pi}\right) \tag{24}$$

$$R_F^{-}\left(v_1 - v_2 - \frac{c}{2},\hat{\pi}\right) L_1^{-}(v_1,\hat{\pi}) L_2^{+}(v_2,\hat{\pi}) = L_2^{+}(v_2,\hat{\pi}) L_1^{-}(v_1,\hat{\pi}) R_F^{*-}\left(v_1 - v_2 + \frac{c}{2},\hat{\pi}\right) \tag{25}$$

$$R_F^{\pm}(v_1 - v_2,\hat{\pi}) L_1^{\pm}(v_1,\hat{\pi}) L_2^{\pm}(v_2,\hat{\pi}) = L_2^{\pm}(v_2,\hat{\pi}) L_1^{\pm}(v_1,\hat{\pi}) R_F^{*\pm}(v_1 - v_2,\hat{\pi}) \tag{26a}$$

$$L^{-}(v,\hat{\pi}) = L^{+}(v - \frac{c}{2} + r,\hat{\pi}) \tag{26b}$$

where $L_1^{\pm}(v,\hat{\pi}) = L^{\pm}(v,\hat{\pi})\otimes\mathrm{id}$, $L_2^{\pm}(v,\hat{\pi}) = \mathrm{id}\otimes L^{\pm}(v,\hat{\pi})$, $R_F^{*\pm}(v,\hat{\pi}) = R_F^{\pm}(v,-\hat{\pi},r-c)$ and c is the centre of the algebra (its value on some representation of the algebra is usually called the level of the algebra). Moreover, the $L^{\pm}$-operators are related with the dynamical variable $\hat{\pi}$:

$$\hat{\pi} L_{\nu}^{(\pm)\mu}(v,\hat{\pi}) = L_{\nu}^{(\pm)\mu}(v,\hat{\pi})(\hat{\pi} + (\nu r - (r-c)\mu)).$$

Hence, the dynamical R-matrices have the following properties

$$R_F^{\epsilon}(v,\hat{\pi}) L_{\nu}^{(\epsilon')\mu}(v,\hat{\pi}) = L_{\nu}^{(\epsilon')\mu}(v,\hat{\pi}) R_F^{\epsilon}(v,\hat{\pi} + \mu c) \tag{27}$$

$$R_F^{*\epsilon}(v,\hat{\pi}) L_{\nu}^{(\epsilon')\mu}(v,\hat{\pi}) = L_{\nu}^{(\epsilon')\mu}(v,\hat{\pi}) R_F^{*\epsilon}(v,\hat{\pi} + \nu c) \tag{28}$$

where $\epsilon, \epsilon' \in \pm$ and the following property

$$[v+t]_t = -[v]$$

is used.

Remark. The relation equation (25) is the direct result of equations (22) and (24). The analytical continuation relation (26b) follows from equation (23b) with $\tau^{\pm}$ independent of r.

Let

$$L^{\pm}(v,\hat{\pi}) = \begin{pmatrix} 1 & 0 \\ E^{\pm}(v) & 1 \end{pmatrix} \begin{pmatrix} K_1^{\pm}(v) & 0 \\ & K_2^{\pm}(v) \end{pmatrix} \begin{pmatrix} 1 & F^{\pm}(v) \\ 0 & 1 \end{pmatrix}$$

be the Gauss decomposition of $L^{\pm}$-operators. For convenience, we introduce the following symbols

$$R_F^{\pm}(v,\hat{\pi}) = \begin{pmatrix} a^{\pm}(v) & & & \\ & b^{\pm}(v) & c^{\pm}(v) & \\ & d^{\pm}(v) & e^{\pm}(v) & \\ & & & a^{\pm}(v) \end{pmatrix}$$

$$R_F^{*\pm}(v,\hat{\pi}) = \begin{pmatrix} a'^{\pm}(v) & & & \\ & b'^{\pm}(v) & c'^{\pm}(v) & \\ & d'^{\pm}(v) & e'^{\pm}(v) & \\ & & & a'^{\pm}(v) \end{pmatrix}.$$

Remark. The elements $a^{\pm}(v)$ and $a'^{\pm}(v)$ do not depend on the dynamical variable, and commute with the Gauss components of $L^{\pm}$-operators.

Define the total currents $E(v)$ and $F(v)$ by using the corresponding Ding–Frenkel correspondence

$$E(v) = E^{+}(v) - E^{-}\left(v + \frac{c}{2}\right) \qquad F(v) = F^{+}\left(v + \frac{c}{2}\right) - F^{-}(v). \tag{29}$$

We then have the following proposition.

Proposition 1. The total currents $E(v), F(v)$ and $K_i^{\pm}(v)$ $(i = 1, 2)$ satisfy the following commutation relations

$$a^{\pm}(v_1 - v_2)K_i^{\pm}(v_1)K_i^{\pm}(v_2) = K_i^{\pm}(v_2)K_i^{\pm}(v_1)a'^{\pm}(v_1 - v_2) \tag{30a}$$

$$b^{\pm}(v_1 - v_2)K_1^{\pm}(v_1)K_2^{\pm}(v_2) = K_2^{\pm}(v_2)K_1^{\pm}(v_1)b'^{\pm}(v_1 - v_2) \tag{30b}$$

$$a^{+}\left(v_1 - v_2 + \frac{c}{2}\right)K_i^{+}(v_1)K_i^{-}(v_2) = K_i^{-}(v_2)K_i^{+}(v_1)a'^{\pm}\left(v_1 - v_2 - \frac{c}{2}\right) \tag{30c}$$

$$b^{+}\left(v_1 - v_2 + \frac{c}{2}\right)K_1^{+}(v_1)K_2^{-}(v_2) = K_2^{-}(v_2)K_1^{+}(v_1)b'^{+}\left(v_1 - v_2 - \frac{c}{2}\right) \tag{30d}$$

$$b^{-}\left(v_1 - v_2 - \frac{c}{2}\right)K_1^{-}(v_1)K_2^{+}(v_2) = K_2^{+}(v_2)K_1^{-}(v_1)b'^{-}\left(v_1 - v_2 + \frac{c}{2}\right) \tag{30e}$$

$$K_1^{+}(v_1)E(v_2)K_1^{+}(v_1)^{-1} = \frac{a^{+}(v_1 - v_2)}{b^{+}(v_1 - v_2)}E(v_2) \tag{31a}$$

$$K_2^{+}(v_2)E(v_1)K_2^{+}(v_2)^{-1} = E(v_1)\frac{a^{+}(v_1 - v_2)}{b^{+}(v_1 - v_2)} \tag{31b}$$

$$K_1^{-}(v_1)E(v_2)K_1^{-}(v_1)^{-1} = \frac{a^{+}(v_1 - v_2 - \frac{c}{2})}{b^{+}(v_1 - v_2 - \frac{c}{2})}E(v_2) \tag{31c}$$

$$K_2^{-}(v_2)E(v_1)K_2^{-}(v_2)^{-1} = E(v_1)\frac{a^{+}(v_1 - v_2 + \frac{c}{2})}{b^{+}(v_1 - v_2 + \frac{c}{2})} \tag{31d}$$

$$K_1^{+}(v_1)^{-1}F(v_2)K_1^{+}(v_1) = F(v_2)\frac{a'^{+}(v_1 - v_2 - \frac{c}{2})}{b'^{+}(v_1 - v_2 - \frac{c}{2})} \tag{32a}$$

$$K_2^{+}(v_2)^{-1}F(v_1)K_2^{+}(v_2) = \frac{a'^{+}(v_1 - v_2 + \frac{c}{2})}{b'^{+}(v_1 - v_2 + \frac{c}{2})}F(v_1) \tag{32b}$$

$$K_1^{-}(v_1)^{-1}F(v_2)K_1^{-}(v_1) = F(v_2)\frac{a'^{+}(v_1 - v_2)}{b'^{+}(v_1 - v_2)} \tag{32c}$$

$$K_2^-(v_2)^{-1}F(v_1)K_2^-(v_2) = \frac{a'^+(v_1-v_2)}{b'^+(v_1-v_2)}F(v_1) \tag{32d}$$

$$E(v_1)\frac{a^\pm(v_1-v_2)}{b^\pm(v_1-v_2)}E(v_2) = E(v_2)\frac{a^\pm(v_2-v_1)}{b^\pm(v_2-v_1)}E(v_1) \tag{33a}$$

$$F(v_1)\frac{a'^\pm(v_2-v_1)}{b'^\pm(v_2-v_1)}F(v_2) = F(v_2)\frac{a'^\pm(v_1-v_2)}{b'^\pm(v_1-v_2)}F(v_1) \tag{33b}$$

$$\begin{aligned}[E(v_1), F(v_2)] = \frac{1}{x-x^{-1}}&\left\{\delta\left(v_2-v_1-\frac{c}{2}\right)K_2^-\left(v_1+\frac{c}{2}\right)\frac{[\hat\pi]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat\pi-1]_{r-c}}K_1^-\left(v_1+\frac{c}{2}\right)^{-1}\right.\\ &\left.-\,\delta\left(v_2-v_1+\frac{c}{2}\right)K_2^+(v_1)\frac{[\hat\pi]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat\pi-1]_{r-c}}K_1^+(v_1)^{-1}\right\}\end{aligned} \tag{34}$$

where $K_i^-(v) = K_i^+(v-\frac{c}{2}+r)$ and $\theta'_t = (x-x^{-1})\frac{\partial}{\partial v}[v]_t|_{v=0}$.

The proof of these relations is in appendix B.

Remark. The elliptic algebra $U_{q,p}(\widehat{sl_2})$ proposed by Konno [27], has different $K(v)$ from our ($K_i^\pm(v)$ also in [37]), in which the commutation relations between $K(v)$ and $E(v)$, $F(v)$ do not depend on the dynamical variable. However, they share the same subalgebra generated by $H^\pm(v)$,$E(v)$ and $F(v)$ (see below).

Set

$$H^+(v) = K_2^-\left(v+\frac{c}{4}\right)\frac{[\hat\pi]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat\pi-1]_{r-c}}K_1^-\left(v+\frac{c}{4}\right)^{-1} \tag{35}$$

$$H^-(v) = K_2^+\left(v+\frac{c}{4}\right)\frac{[\hat\pi]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat\pi-1]_{r-c}}K_1^+\left(v+\frac{c}{4}\right)^{-1}. \tag{36}$$

Also in the case of q-affine algebra [7] and Yangian double algebra [8, 9], we can obtain the corresponding Drinfeld current algebra of $A_{q,p;\hat\pi}(\widehat{gl_2})$ which is the subalgebra of the elliptic algebra $A_{q,p;\hat\pi}(\widehat{gl_2})$ and generated by $E(v)$, $F(v)$, $H^\pm(v)$. We then find the following proposition.

Proposition 2. The ellipitc (Drinfeld) current algebra of algebra $A_{q,p;\hat\pi}(\widehat{gl_2})$ is generated by $E(v)$, $F(v)$, $H^\pm(v)$ with the following algebraic relations

$$E(v_1)E(v_2) = \frac{[v_1-v_2-1]_r}{[v_1-v_2+1]_r}E(v_2)E(v_1) \tag{37}$$

$$F(v_1)F(v_2) = \frac{[v_1-v_2+1]_{r-c}}{[v_1-v_2-1]_{r-c}}F(v_2)F(v_1) \tag{38}$$

$$\begin{aligned}[E(v_1), F(v_2)] = \frac{1}{x-x^{-1}}&\left\{\delta\left(v_1-v_2+\frac{c}{2}\right)H^+\left(v_1+\frac{c}{4}\right)\right.\\ &\left.-\,\delta\left(v_1-v_2-\frac{c}{2}\right)H^-\left(v_1-\frac{c}{4}\right)\right\}\end{aligned} \tag{39}$$

$$H^\pm(v_1)E(v_2) = \frac{[v_1-v_2-1\mp\frac{c}{4}]_r}{[v_1-v_2+1\mp\frac{c}{4}]_r}E(v_2)H^\pm(v_1) \tag{40}$$

$$H^\pm(v_1)F(v_2) = \frac{[v_1-v_2+1\pm\frac{c}{4}]_{r-c}}{[v_1-v_2-1\pm\frac{c}{4}]_{r-c}}F(v_2)H^\pm(v_1) \tag{41}$$

$$H^\pm(v_1)H^\pm(v_2) = \frac{[v_1-v_2+1]_{r-c}[v_1-v_2-1]_r}{[v_1-v_2-1]_{r-c}[v_1-v_2+1]_r}H^\pm(v_2)H^\pm(v_1) \tag{42}$$

$$H^+(v_1)H^-(v_2) = \frac{[v_1 - v_2 + 1 + \frac{c}{2}]_{r-c}[v_1 - v_2 - 1 - \frac{c}{2}]_r}{[v_1 - v_2 - 1 + \frac{c}{2}]_{r-c}[v_1 - v_2 + 1 - \frac{c}{2}]_r} H^-(v_2)H^+(v_1) \tag{43}$$

and

$$H^-(v) = H^+\left(v + \frac{c}{2} - r\right).$$

The proof of these formulae is in appendix C.

Remark.

(1) The deformed parameters are $q = x$ and $p = x^{2r}$ (cf [10]). Moreover, the constructure coefficients in equations (37)–(43) do not depend on the dynamical variable $\hat{\pi}$.

(2) When $r \longrightarrow +\infty$, the limit current algebra is the algebra $U_q(\widehat{sl_2})$ [32].

One can see that if $c = 1$ (i.e. level one), the current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ are the algebra of screening currents for q-Virasoro algebra (cf equations (9)–(15)) which play the role of symmetry algebra in the ABF model [12, 17]. For the general level k ($k \in$ integer), the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ would correspond to the k-fusion ABF model [26, 27, 29] and in this case, some q-deformation of the extended Virasoro algebra [38–40] would exist in such a way that their screening currents satisfy the current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. So, this elliptic algebra would play an important role in the studies of $A_1^{(1)}$-type face models as that of the algebra $A_{q,p}(\widehat{gl_2})$ in the eight-vertex model [10].

3.3. The algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ as the dynamically twisted algebra $A_{q,p}(\widehat{gl_2})$

It is well known that there exists a face-vertex correspondence between $A_1^{(1)}$ face model and eight-vertex model when r is a generic one [1, 26, 28, 29].This would result in the ‘equivalence’ between the underlying algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and $A_{q,p}(\widehat{sl_2})$—the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ is the dynamcially twisted algebra of $A_{q,p}(\widehat{gl_2})$. We are informed that Jimbo had some similar ideal to use the face-vertex correspondence at the level of vertex operators. In this section, we restrict our attention to the case of r being a generic one.

Let ϵ_μ ($\mu \in \pm$) be the orthonormal basis in R^2, which are supplied with the inner product $\langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu\nu}$. Set

$$\overline{\epsilon}_\mu = \epsilon_\mu - \epsilon \qquad \epsilon = \frac{\epsilon_- + \epsilon_+}{2}.$$

Then, define the intertwiners [28, 33]

$$\begin{aligned}
&\varphi_{\hat{k},\mu}^{(m)}(v) = \theta^{(m)}\left(\frac{v + \langle \hat{k}, \overline{\epsilon}_\mu \rangle}{r}, -\frac{1}{rw}\right) \\
&\varphi_{\mu,\hat{l}}^{\prime(m)}(v) = \theta^{(m)}\left(\frac{v + \langle \hat{l}, \overline{\epsilon}_\mu \rangle}{r - c}, -\frac{1}{(r-c)w}\right) \\
&\langle \hat{k}, \overline{\epsilon}_\mu \rangle \equiv \mu\hat{k} \qquad \langle \hat{l}, \overline{\epsilon}_\mu \rangle \equiv \mu\hat{l} \\
&\hat{\pi} \equiv (r-c)\hat{k} - r\hat{l}.
\end{aligned} \tag{43a}$$

Here, we remark that the decomposition of equation (43a) can be defined only for generic r [33] and

$$\hat{k}L^{(\pm)\mu}{}_\nu(v, \hat{\pi}) = L^{(\pm)\mu}{}_\nu(v, \hat{\pi})(\hat{k} + \mu c) \qquad \hat{l}L^{(\pm)\mu}{}_\nu(v, \hat{\pi}) = L^{(\pm)\mu}{}_\nu(v, \hat{\pi})(\hat{l} + \nu c).$$

The face-vertex correspondence relations read as

$$R^{ij}_{mn}(v_1-v_2)\varphi^{(m)}_{\hat{k},\nu}(v_1)\varphi^{(n)}_{\hat{k}-\overline{\epsilon}_\nu,\mu}(v_2)=\sum_{\mu'\nu'}R^{\nu\mu}_{F\nu'\mu'}(v_1-v_2,\hat{\pi})\varphi^{(i)}_{\hat{k}-\overline{\epsilon}_{\mu'},\nu'}(v_1)\varphi^{(j)}_{\hat{k},\mu'}(v_2)$$

$$R^{*ij}_{mn}(v_1-v_2)\varphi'^{(m)}_{\nu,\hat{l}+\overline{\epsilon}_\mu}(v_1)\varphi'^{(n)}_{\mu,\hat{l}}(v_2)=\sum_{\mu\nu}R^{*\nu\mu}_{F\nu'\mu'}(v_1-v_2,\hat{\pi})\varphi'^{(i)}_{\nu',\hat{l}}(v_1)\varphi'^{(j)}_{\mu',\hat{l}+\overline{\epsilon}_{\nu'}}(v_2)$$

where the nondynamical R-matrices R and R^* are the same as that of Foda *et al* [10]. Moreover, we can introduce intertwiners $\overline{\varphi}_{\hat{k},\mu}$ and $\overline{\varphi}'_{\mu,\hat{l}}$ satisfying relations [28]

$$\sum_m \overline{\varphi}^{(m)}_{\mu,\hat{k}}\varphi^{(m)}_{\nu,\hat{k}}=\delta_{\mu,\nu}\qquad \sum_\mu \overline{\varphi}^{(i)}_{\mu,\hat{k}}\varphi^{(j)}_{\mu,\hat{k}}=\delta^{ij}$$

$$\sum_m \overline{\varphi}'^{(m)}_{\hat{l},\mu}\varphi'^{(m)}_{\hat{l},\nu}=\delta_{\mu,\nu}\qquad \sum_\mu \overline{\varphi}'^{(i)}_{\hat{l},\mu}\varphi'^{(j)}_{\hat{l},\mu}=\delta^{ij}.$$

We then have the twisted relations between the R-matrix of eight-vertex model and the R-matrix of $A^{(1)}_1$ face model

$$R^{\nu\mu}_{F\nu'\mu'}(v_1-v_2,\hat{\pi})=\sum_{ijmn}\overline{\varphi}^{(j)}_{\hat{k},\mu'}(v_2)\overline{\varphi}^{(i)}_{\hat{k}-\overline{\epsilon}_{\mu'},\nu'}(v_1)R^{ij}_{mn}(v_1-v_2)\varphi^{(m)}_{\hat{k},\nu}(v_1)\varphi^{(n)}_{\hat{k}-\overline{\epsilon}_\nu,\mu}(v_2) \tag{44}$$

$$R^{*\nu\mu}_{F\nu'\mu'}(v_1-v_2,\hat{\pi})=\sum_{ijmn}\overline{\varphi}'^{(j)}_{\mu',\hat{l}+\overline{\epsilon}_{\nu'}}(v_2)\overline{\varphi}'^{(i)}_{\nu',\hat{l}}(v_1)R^{*ij}_{mn}(v_1-v_2)\varphi'^{(m)}_{\nu,\hat{l}+\overline{\epsilon}_\mu}(v_1)\varphi'^{(n)}_{\mu,\hat{l}}(v_2). \tag{45}$$

Moreover, we can construct the twisted relations between the corresponding $L^{\pm}$-operators

$$\begin{aligned} L^{\pm\nu}_{\mu}(v,\hat{\pi})&=\sum_{mm'}\overline{\varphi}^{(m')}_{\hat{k},\nu}(v)L^{\pm m'}_{m}(v)\varphi'^{(m)}_{\mu,\hat{l}}(v)\\ L^{\pm m'}_{m}(v)&=\sum_{\mu\nu}\varphi^{(m')}_{\hat{k},\nu}(v)L^{\pm\nu}_{\mu}(v,\hat{\pi})\overline{\varphi}'^{(m)}_{\mu,\hat{l}}(v). \end{aligned} \tag{46}$$

We then have the following proposition.

Proposition 3. The $L^{\pm}(v)$ operators given by the twisted relations equation (46) satisfying the commutation relations of the algebra $A_{q,p}(\widehat{gl_2})$ [10]

$$R^{+}\left(v_1-v_2+\frac{c}{2}\right)L^{+}_1(v_1)L^{-}_2(v_2)=L^{-}_2(v_2)L^{+}_1(v_1)R^{*+}\left(v_1-v_2-\frac{c}{2}\right) \tag{47}$$

$$R^{\pm}(v_1-v_2)L^{\pm}_1(v_1)L^{\pm}_2(v_2)=L^{\pm}_2(v_2)L^{\pm}_1(v_1)R^{*\pm}(v_1-v_2) \tag{48}$$

where the r-matrices $R^{\pm}(v)$, $R^{*\pm}(v)$ are the same as that of Foda *et al*

$$R^{\pm}(v)\equiv\tau^{\pm}(v)R(v)\qquad R^{*\pm}(v)=R^{*\pm}(v)|_{r\longrightarrow r-c}.$$

4. The type I and type II vertex and Miki's construction

This section is devoted to the realization of infinite-dimensional representations of the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one by the q-primary fields of q-Virasoro algebra.

4.1. The type I and type II vertex operators

The method of bosonization provides a powerful method to study the solvable lattice model in both the vertex-type model [32] and the face-type model [17, 30, 35]. In this section, we give the bosonization of the type I [17] and type II [35] vertex operator in the ABF model by one free field.

The type I vertex operator corresponds to the half-column transfer matrix of the model, and the type II vertex operator is expected to create the eigenstates of the transfer matrix. We denote the two types of vertex operator as

- vertex operator of type I :$\Phi_i(v)$
- vertex operator of type II:$\Psi^*_i(v)$.

These vertex operators realize the Faddeev–Zamolodchikov (ZF) algebra with a dynamical R-matrix as its structure coefficients

$$\Phi_\nu(v_2)\Phi_\mu(v_1) = R^{\mu'\nu'}_{F\mu\nu}(v_1 - v_2, \hat{\pi})\Phi_{\mu'}(v_1)\Phi_{\nu'}(v_2) \tag{49}$$

$$\Psi^*_\mu(v_1)\Psi^*_\nu(v_2) = -R^{*\mu\nu}_{F\mu'\nu'}(v_1 - v_2, \hat{\pi})\Psi^*_{\nu'}(v_2)\Psi^*_{\mu'}(v_1) \tag{50}$$

$$\Phi_\nu(v_1)\Psi^*_\mu(v_2) = \tau(v_1 - v_2)\Psi^*_\mu(v_2)\Phi_\nu(v_1). \tag{51}$$

Let us introduce the other basic operators

$$\eta_1(v) = \mathrm{e}^{-\mathrm{i}\sqrt{\frac{r-1}{r}}(Q-\mathrm{i}2v\ln xP)} : \mathrm{e}^{-\sum_{m\neq 0}\frac{\beta_m}{m}x^{-2vm}} :$$

$$\eta'_1(v) = \mathrm{e}^{\mathrm{i}\sqrt{\frac{r}{r-1}}(Q-\mathrm{i}2v\ln xP)} : \mathrm{e}^{\sum_{m\neq 0}\frac{\beta'_m}{m}x^{-2vm}} :$$

$$\xi(v) = \mathrm{e}^{\mathrm{i}\sqrt{\frac{2(r-1)}{r}}(Q-\mathrm{i}2v\ln xP)} : \mathrm{e}^{\sum_{m\neq 0}\frac{x^m+x^{-m}}{m}\beta_m x^{-2vm}} :$$

$$\xi'(v) = \mathrm{e}^{-\mathrm{i}\sqrt{\frac{2r}{r-1}}(Q-\mathrm{i}2v\ln xP)} : \mathrm{e}^{-\sum_{m\neq 0}\frac{x^m+x^{-m}}{m}\beta'_m x^{-2vm}} :$$

where the q-deformed bosonic oscillators β_m, P, Q are defined in equation (3). Then, the bosonization of vertex operators is given by [17, 30, 33]

$$\Phi_+(v) = \eta_1(v) \qquad \Phi_-(v) = \oint_C \frac{\mathrm{d}(x^{2v_1})}{2\pi\mathrm{i}x^{2v_1}}\eta_1(v)\xi(v_1)f(v_1 - v, \hat{\pi}) \tag{52}$$

$$\Psi_+(v) = \eta'_1(v) \qquad \Psi_-(v) = \oint_{C'} \frac{\mathrm{d}(x^{2v_1})}{2\pi\mathrm{i}x^{2v_1}}\eta'_1(v)\xi'(v_1)f'(v_1 - v, \hat{\pi}) \tag{53}$$

where the integration contour C is a simple closed curve around the origin satisfying $|xx^{2v}| < |x^{2v_1}| < |x^{-1}x^{2v}|$; C' is chosen in such a way that the poles $x^{2v-1+2n(r-1)}$ $(0 \leqslant n)$ are inside and the poles $x^{2v+1-2n(r-1)}$ $(0 \leqslant n)$ are outside, and

$$f(v, w) = \frac{[v + \frac{1}{2} - w]_r}{[v - \frac{1}{2}]_r} \qquad f'(v, w) = \frac{[v - \frac{1}{2} + w]_{r-1}}{[v + \frac{1}{2}]_{r-1}}.$$

Set

$$\Psi^*_\mu(v) = \Psi_{-\mu}(v)\frac{1}{[\hat{\pi}]_{r-1}}.$$

From the normal order relations given in appendix A, one can check that the bosonic realization for Φ_i and Ψ^* in equations (52) and (53) satisfy the ZF algebra in equations (49)–(51) [30, 33].

4.2. The realization of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one by Miki's construction

Let us introduce Miki's construction [34]

$$L^{+\mu}_\nu(v, \hat{\pi}) = \Phi_\mu(v)\Psi^*_\nu(v - \tfrac{1}{2}) \tag{54}$$

$$L^{-\mu}_\nu(v, \hat{\pi}) = \Phi_\mu(v - \tfrac{1}{2})\Psi^*_\nu(v). \tag{55}$$

Using the relations of the ZF algebra in equations (49)–(51), one can prove that the $L^\pm$-operators constructed above satisfy the definition of the elliptic quantum algebra of equations (24)–(26) with $c = 1$. Moreover, we have the following proposition.

Proposition 4. The $L^{\pm}$-operators and two-type vertex operators satisfy the folllowing relations

$$R_F^+(v_1 - v_2, \hat{\pi}) L_1^+(v_1, \hat{\pi}) \Phi_2(v_2) = \Phi_2(v_2) L_1^+(v_1, \hat{\pi}) \tag{56}$$

$$R_F^-(v_1 - v_2 - \tfrac{1}{2}, \hat{\pi}) L_1^-(v_1, \hat{\pi}) \Phi_2(v_2) = \Phi_2(v_2) L_1^-(v_1, \hat{\pi}) \tag{57}$$

$$L_1^+(v_1, \hat{\pi}) \Psi_2^*(v_2) = \Psi_2^*(v_2) L_1^+(v_1, \hat{\pi}) R_F^{*+}(v_1 - v_2 - \tfrac{1}{2}, \hat{\pi}) \tag{58}$$

$$L_1^-(v_1, \hat{\pi}) \Psi_2^*(v_2) = \Psi_2^*(v_2) L_1^-(v_1, \hat{\pi}) R_F^{*+}(v_1 - v_2, \hat{\pi}). \tag{59}$$

The proof is direct by using Miki's construction of $L^{\pm}$-operators and the ZF algebra of equations (49)–(51).

From proposition 4, one can see that the vertex operators of the ABF model are the intertwining operators of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one, which satisfy some generalized relations of q-affine algebra and its intertwining operators [31].

5. The scaling limit algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$

Another deformed Virasoro algbera—$\hbar$-Virasoro algebra can be considered as symmetries of the massive integrable field theories [14], and at the semiclassical level corresponds to the centre of the Yangian double with centre $DY_{\hbar}(\widehat{gl_2})$ at the critical level [18]. In another way, $\hbar$-Virasoro algbera can be considered as the scaling limit of the q-Virasoro algebra [14]

$$x^{2v} = p^{-\frac{i\beta}{\hbar}} \qquad q = p^{-\frac{1}{\eta}} p \longrightarrow 1.$$

Moreover, the screening currents of $\hbar$-Virasoro algebra satisfies a closed algebra relation, which can also be considered as the scaling limit of that of q-Virasoro algbera in equations (9)–(15) [14]. Therefore, we can construct a generalizing algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ as the scaling limit of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$, which is expected to be the symmetric algebra of the k-fused restricted sine-Gordon model and its (Drinfeld) current algebra would be the algebra of screening currents of some $\hbar$-deformed extended Virasoro algbera. Similarly, the algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ can be formulated in the dynamical $RLL = LLR^*$ form with the dynamical R-matrix being the trigonometric solution to the dynamical Yang–Baxter equation [14].

In this section, we restrict ourselves to the trigonometric dynamical R-matrix (or the scaling limit of the R-matrix in equations (16) and (20)). To avoid confusion with that in the former section, we choose the same symbols as in section 3.1

$$R_F(v, \hat{\pi}) \equiv R_F(v, \hat{\pi}, \eta) = \begin{pmatrix} a & & & \\ & b & c & \\ & d & e & \\ & & & a \end{pmatrix} \tag{60}$$

and

$$a(\beta, \hat{\pi}) = \kappa(\beta) = \exp\left\{ \int_0^\infty \frac{2\,\mathrm{sh}\,\frac{\hbar t}{2}\,\mathrm{sh}\,\frac{\hbar t}{2\eta}\,\mathrm{sh}\,\mathrm{i}\beta t}{\mathrm{sh}\,\hbar t\,\mathrm{sh}\,\frac{(1+\eta)\hbar t}{2\eta}} \frac{\mathrm{d}t}{t} \right\} \tag{61}$$

$$\frac{b(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar}) \sin\pi\eta(\hat{\pi} - 1)}{\sin\pi\eta(\hat{\pi}) \sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar} + 1)} \qquad \frac{c(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta \sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar} + \hat{\pi})}{\sin\pi\eta(\hat{\pi}) \sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar} + 1)} \tag{62}$$

$$\frac{d(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta \sin\pi\eta(-\frac{\mathrm{i}\beta}{\hbar} + \hat{\pi})}{\sin\pi\eta(\hat{\pi}) \sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar} + 1)} \qquad \frac{e(\beta, \hat{\pi})}{a(\beta, \hat{\pi})} = \frac{\sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar}) \sin\pi\eta(\hat{\pi} + 1)}{\sin\pi\eta(\hat{\pi}) \sin\pi\eta(\frac{\mathrm{i}\beta}{\hbar} + 1)} \tag{63}$$

$$R_F^{\pm}(\beta,\hat{\pi}) = \tau^{\pm}(\beta)R_F(\beta,\hat{\pi}) \qquad \tau^{+}(\beta) = \mathrm{ctg}\left(\frac{\mathrm{i}\pi\beta}{2\hbar}\right) \qquad \tau^{-}(\beta) = -\,\mathrm{tg}\left(\frac{\mathrm{i}\pi\beta}{2\hbar}\right).$$

Define

$$R_F^{*\pm}(\beta,\hat{\pi}) \equiv R_F^{\pm}(\beta,-\hat{\pi})|_{\eta\longrightarrow\eta'} \qquad \frac{1}{\eta'} = \frac{1}{\eta} - c.$$

The algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ is generated by the matrix elements of one $L^{\pm}$-operator which satisfy the following relations

$$R_F^{+}\left(\beta_1-\beta_2-\frac{\mathrm{i}\hbar c}{2},\hat{\pi}\right)L_1^{+}(\beta_1,\hat{\pi})L_2^{-}(\beta_2,\hat{\pi}) \\ = L_2^{-}(\beta_2,\hat{\pi})L_1^{+}(\beta_1,\hat{\pi})R_F^{*+}\left(\beta_1-\beta_2+\frac{\mathrm{i}\hbar c}{2},\hat{\pi}\right) \tag{64}$$

$$R_F^{-}\left(\beta_1-\beta_2+\frac{\mathrm{i}\hbar c}{2},\hat{\pi}\right)L_1^{-}(\beta_1,\hat{\pi})L_2^{+}(\beta_2,\hat{\pi}) \\ = L_2^{+}(\beta_2,\hat{\pi})L_1^{-}(\beta_1,\hat{\pi})R_F^{*-}\left(\beta_1-\beta_2-\frac{\mathrm{i}\hbar c}{2},\hat{\pi}\right) \tag{65}$$

$$R_F^{\pm}(\beta_1-\beta_2,\hat{\pi})L_1^{\pm}(\beta_1,\hat{\pi})L_2^{\pm}(\beta_2,\hat{\pi}) = L_2^{\pm}(\beta_2,\hat{\pi})L_1^{\pm}(\beta_1,\hat{\pi})R_F^{*\pm}(\beta_1-\beta_2,\hat{\pi}) \tag{66}$$

$$\hat{\pi}L_{\nu}^{(\pm)\mu}(\beta,\hat{\pi}) = L_{\nu}^{(\pm)\mu}(\beta,\hat{\pi})\left(\hat{\pi}+\left(\nu\frac{1}{\eta}-\frac{1}{\eta'}\mu\right)\right). \tag{67}$$

Let

$$L^{\pm}(\beta,\hat{\pi}) = \begin{pmatrix} 1 & 0 \\ E^{\pm}(\beta) & 1 \end{pmatrix}\begin{pmatrix} K_1^{\pm}(\beta) & 0 \\ & K_2^{\pm}(\beta) \end{pmatrix}\begin{pmatrix} 1 & F^{\pm}(\beta) \\ 0 & 1 \end{pmatrix}$$

be the Gauss decomposition of the $L^{\pm}$-operators. For convenience, we also introduce the following symbols

$$R_F^{\pm}(\beta,\hat{\pi}) = \begin{pmatrix} a^{\pm}(\beta) & & & \\ & b^{\pm}(\beta) & c^{\pm}(\beta) & \\ & d^{\pm}(\beta) & e^{\pm}(\beta) & \\ & & & a^{\pm}(\beta) \end{pmatrix}$$

$$R_F^{*\pm}(\beta,\hat{\pi}) = \begin{pmatrix} a'^{\pm}(\beta) & & & \\ & b'^{\pm}(\beta) & c'^{\pm}(\beta) & \\ & d'^{\pm}(\beta) & e'^{\pm}(\beta) & \\ & & & a'^{\pm}(\beta) \end{pmatrix}.$$

Define the total currents $E(\beta)$ and $F(\beta)$ by the corresponding Ding–Frenkel correspondence

$$E(\beta) = E^{+}(\beta) - E^{-}\left(\beta-\frac{\mathrm{i}\hbar c}{2}\right) \qquad F(\beta) = F^{+}\left(\beta-\frac{\mathrm{i}\hbar c}{2}\right) - F^{-}(\beta). \tag{68}$$

Substituting the Gauss decomposition of the $L^{\pm}$-operator, we can obtain similar commutation relations bewteen $E(\beta)$, $F(\beta)$ and $K_i^{\pm}(\beta)$ as those in proposition 1, where the matrix elements of the R-matrix are in equations (60)–(63).

The algebra $A_{\hbar,\eta}(\widehat{gl_2})$ as the scaling limit of the elliptic algebra $A_{q,p}(\widehat{gl_2})$ was studied by Khoroshkin *et al* through the method of Gauss decomposition [8]. Commutation relations of $E(\beta)$,$F(\beta)$ and $K_i^{\pm}(\beta)$ were obtained, which are quite different from ours. This is due to the fact that they chose a nondynamical '$RLL = LLR^*$' formalism, and consequently, the commutation relations obtained do not depend on the dynamical variable. If we introduce $H^{\pm}(\beta)$ to equations (69) and (70), which are quite different from that of Khoroshkin *et al* algebra $A_{\hbar,\eta;\hat{\pi}}(\widehat{gl_2})$ and algebra $A_{\hbar,\eta}(\widehat{gl_2})$ share the same subalgebra—the (Drinfeld) current algebra of each one is generated by $E(\beta)$, $F(\beta)$, $H^{\pm}(\beta)$. Although they share the same subalgebraic commutation relations, they relate to different E, F, $K_i^{\pm}$ (or different algebra) and consequently are associated with different vertex operators [18, 14] which have different relations to equations (56)–(59). Moreover, the two algebras are related to the different models (face-type model for the dynamical algebra and vertex model for the nondynamical algebra) especially for the rational η case.

Setting

$$H^{+}(\beta) = K_2^{-}\left(\beta - \frac{\mathrm{i}\bar{h}c}{4}\right)\frac{\sin\pi\eta'\hat{\pi}\,\sin\pi\eta'}{\pi\eta'\sin\pi\eta'(\hat{\pi}-1)}K_1^{-}\left(\beta - \frac{\mathrm{i}\bar{h}c}{4}\right)^{-1} \tag{69}$$

$$H^{-}(\beta) = K_2^{+}\left(\beta - \frac{\mathrm{i}\bar{h}c}{4}\right)\frac{\sin\pi\eta'\hat{\pi}\,\sin\pi\eta'}{\pi\eta'\sin\pi\eta'(\hat{\pi}-1)}K_1^{+}\left(\beta - \frac{\mathrm{i}\bar{h}c}{4}\right)^{-1} \tag{70}$$

we have the current algebra of algebra $A_{\bar{h},\eta;\hat{\pi}}(\widehat{gl_2})$ generated by $E(\beta)$, $F(\beta)$ and $H^{\pm}(\beta)$ which is the scaling limit of those of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$.

Proposition 5. The current algebra of algebra $A_{\bar{h},\eta;\hat{\pi}}(\widehat{gl_2})$ is generated by $E(\beta)$, $F(\beta)$, $H^{\pm}(\beta)$ and satisfies the following relations

$$E(\beta_1)E(\beta_2) = \frac{\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h})}{\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h})}E(\beta_2)E(\beta_1) \tag{71}$$

$$F(\beta_1)F(\beta_2) = \frac{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h})}{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h})}F(\beta_2)F(\beta_1) \tag{72}$$

$$\begin{aligned}[E(\beta_1), F(\beta_2)] = \bar{h}\Bigg\{&\delta\left(\beta_1-\beta_2-\frac{\mathrm{i}\bar{h}c}{2}\right)H^{+}\left(\beta_1-\frac{\mathrm{i}\bar{h}c}{4}\right)\\ &-\delta\left(\beta_1-\beta_2+\frac{\mathrm{i}\bar{h}c}{2}\right)H^{-}\left(\beta_1+\frac{\mathrm{i}\bar{h}c}{4}\right)\Bigg\}\end{aligned} \tag{73}$$

$$H^{\pm}(\beta_1)E(\beta_2) = \frac{\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h}\mp\frac{\mathrm{i}\bar{h}c}{4})}{\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h}\mp\frac{\mathrm{i}\bar{h}c}{4})}E(\beta_2)H^{\pm}(\beta_1) \tag{74}$$

$$H^{\pm}(\beta_1)F(\beta_2) = \frac{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h}\pm\frac{\mathrm{i}\bar{h}c}{4})}{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h}\pm\frac{\mathrm{i}\bar{h}c}{4})}F(\beta_2)H^{\pm}(\beta_1) \tag{75}$$

$$H^{\pm}(\beta_1)H^{\pm}(\beta_2) = \frac{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h})\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h})}{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h})\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h})}H^{\pm}(\beta_2)H^{\pm}(\beta_1) \tag{76}$$

$$H^{+}(\beta_1)H^{-}(\beta_2) = \frac{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h}-\frac{\mathrm{i}\bar{h}c}{2})\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h}+\frac{\mathrm{i}\bar{h}c}{2})}{\sin\frac{\mathrm{i}\pi\eta'}{\bar{h}}(\beta_1-\beta_2-\mathrm{i}\bar{h}-\frac{\mathrm{i}\bar{h}c}{2})\sin\frac{\mathrm{i}\pi\eta}{\bar{h}}(\beta_1-\beta_2+\mathrm{i}\bar{h}+\frac{\mathrm{i}\bar{h}c}{2})}H^{-}(\beta_2)H^{+}(\beta_1). \tag{77}$$

It can be seen that when $c = 1$ (i.e. at level one), the current algebra of algebra $A_{\bar{h},\eta;\hat{\pi}}(\widehat{gl_2})$ is the algebra of the screening currents for $\bar{h}$-Virasoro algebra [14]. For a higher level, it would be the algebra of screening currents for $\bar{h}$-deformed extended Virasoro algebra. Moreover, there exist the following relations between the algebra $A_{\bar{h},\eta;\hat{\pi}}(\widehat{gl_2})$ and $A_{\bar{h},\eta}(\widehat{gl_2})$ [8], and between the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ and $A_{q,p}(\widehat{gl_2})$ [10] for the generic r and η case

$$\begin{array}{ccc} A_{q,p;\hat{\pi}}(\widehat{gl_2}) & \xrightarrow{\text{scaling limit}} & A_{\bar{h},\eta;\hat{\pi}}(\widehat{gl_2}) \\ \updownarrow \text{twisted} & & \updownarrow \text{twisted} \\ A_{q,p}(\widehat{gl_2}) & \xrightarrow{\text{scaling limit}} & A_{\bar{h},\eta}(\widehat{gl_2}). \end{array}$$

6. Discussion

In this paper, we propose an elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ based on a dynamical relations $RLL = LLR^*$, where the dynamical R-matrix is of the $A_1^{(1)}$-type face model. The corresponding (Drinfeld) current algebra is the current algebra generalizing the screening currents for q-Viarsoro algebra and is a dynamical twisted algebra of $A_{q,p}(\widehat{gl_2})$, which can be considered as the results of the correspondence between the $A_1^{(1)}$ face model and the eight-vertex model with generic r. Moreover, the q-primary fields of q-Virasoro Φ_μ and Ψ_μ^* which are also known as the vertex operators for $A_1^{(1)}$ face model, are the intertwining operators of the elliptic algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ at level one.

It is very interesting to investigate the quasi-Hopf structure of the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. Recent work of Jimbo *et al* [45] shows that the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ (i.e. the algebra $B_{q,\lambda}(\widehat{gl_2})$ in [45]) can be obtained by twisting the standard quantum affine algebra $U_q(\widehat{gl_2})$ [45, 46]. This means that the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$ could be endowed with quasi-Hopf structure. It also becomes clear that the evaluation representation of the algebra (or level zero representation) is the representation of Felder *et al* [48].

The work of Jimbo *et al* [47] shows that the 'half-currents' $E^{\pm}(v), F^{\pm}(v)$ can be reconstructed as certain contour integrals of the 'total currents' $E(v), F(v)$ involving some theta-functional factor depending upon the zero-mode (a similar construction was also proposed by Enriquez *et al* [22]). This gives one some hint as to why the Lukyanov's screening operator [17, 20, 21] will include some theta-functional factor.

In this paper, we only consider the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$, but not the algebra $A_{q,p;\hat{\pi}}(\widehat{sl_2})$. Traditionally, one can impose some quantum determinant condition on the case of gl_2 and obtain the corresponding algebra of the sl_2 case. Unfortunately, we still cannot find the corresponding quantum determinant for the algebra $A_{q,p;\hat{\pi}}(\widehat{gl_2})$. However, the subalgebra generated by $E(v)$, $F(v)$, $H^{\pm}(v)$ (in proposition 2) is the algebra $A_{q,p;\hat{\pi}}(\widehat{sl_2})$. So, the vertex operators in section 4 are actually the vertex operators of the algebra $A_{q,p;\hat{\pi}}(\widehat{sl_2})$.

It is very interesting to extend the present formulation $RLL = LLR^*$ to the case of $A_{n-1}^{(1)}$. The corresponding elliptic algebra is $A_{q,p;\hat{\pi}}(\widehat{gl_n})$. The corresponding (Drinfeld) current algebra of algebra $A_{q,p;\hat{\pi}}(\widehat{gl_n})$ would be the current algebra generalizing, the screening currents for q-deformed W_n algebra, which is generated by $E_j(v)$, $F_j(v)$ and $H_j^{\pm}(v)$ $(j = 1, \ldots, n-1)$ with the following relations

$$E_i(v_1)E_j(v_2) = (-1)^{A_{ij}} \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]_r}{[v_1 - v_2 + \frac{A_{ij}}{2}]_r} E_j(v_2)E_i(v_1)$$

$$F_i(v_1)F_j(v_2) = (-1)^{A_{ij}} \frac{[v_1 - v_2 + \frac{A_{ij}}{2}]_{r-c}}{[v_1 - v_2 - \frac{A_{ij}}{2}]_{r-c}} F_j(v_2)F_i(v_1)$$

$$\begin{aligned}[E_j(v_1), F_j(v_2)] = \frac{1}{x - x^{-1}} \Big\{ &\delta\left(v_1 - v_2 + \frac{c}{2}\right) H_j^+\left(v_1 + \frac{c}{4}\right) \\ &- \delta\left(v_1 - v_2 - \frac{c}{2}\right) H_j^-\left(v_1 - \frac{c}{4}\right)\Big\}\end{aligned}$$

$$E_j(v_1)F_{j+1}(v_2) = -F_{j+1}(v_2)E_j(v_1) \qquad F_j(v_1)E_{j+1}(v_2) = -E_{j+1}(v_2)F_j(v_1)$$

$$[E_j(v_1), F_l(v_2)] = 0 \qquad |j - l| > 1$$

$$H_i^{\pm}(v_1)E_j(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2} \mp \frac{c}{4}]_r}{[v_1 - v_2 + \frac{A_{ij}}{2} \mp \frac{c}{4}]_r} E_j(v_2)H_j^{\pm}(v_1)$$

$$H_i^{\pm}(v_1)F_j(v_2) = \frac{[v_1 - v_2 + \frac{A_{ij}}{2} \pm \frac{c}{4}]_{r-c}}{[v_1 - v_2 - \frac{A_{ij}}{2} \pm \frac{c}{4}]_{r-c}} F_j(v_2)H_j^{\pm}(v_1)$$

$$H_i^{\pm}(v_1)H_j^{\pm}(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2}]_r[v_1 - v_2 + \frac{A_{ij}}{2}]_{r-c}}{[v_1 - v_2 + \frac{A_{ij}}{2}]_r[v_1 - v_2 - \frac{A_{ij}}{2}]_{r-c}} H_j^{\pm}(v_2)H_i^{\pm}(v_1)$$

$$H_i^{\pm}(v_1)H_j^{\mp}(v_2) = \frac{[v_1 - v_2 - \frac{A_{ij}}{2} \mp \frac{c}{2}]_r[v_1 - v_2 + \frac{A_{ij}}{2} \pm \frac{c}{2}]_{r-c}}{[v_1 - v_2 + \frac{A_{ij}}{2} \mp \frac{c}{2}]_r[v_1 - v_2 - \frac{A_{ij}}{2} \pm \frac{c}{2}]_{r-c}} H_j^{\mp}(v_2)H_i^{\pm}(v_1)$$

where $H_j^-(v) = H_j^+(v + \frac{c}{2} - r)$ and the matrix A_{ij} is the Cartan matrix for $A_{n-1}^{(1)}$ Lie algebra

$$A_{ij} = 2\delta_{ij} - \delta_{i+1,j} - \delta_{i-1,j}.$$

The above algberaic relations could be derived by the Gauss decomposition of the $L^{\pm}$-operators corresponding to the dynamical R-matrix of $A_{n-1}^{(1)}$ face model. We will present the results in a further paper.

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Note added in proof. After our paper was submitted to the electronic archive, Professor H Konno informed us that 'Recently, Jimbo has succeeded in deriving the algebra $U_{q,p}(\widehat{sl_2})$ by the Gauss decomposition of a central extended dynamical RLL—relations with the R-matrix introduced by Enriquez and Felder. These results allow us to clarify a Hopf algebra structure of $U_{q,p}(\widehat{sl_2})$. Work along these lines is now in progress'. Moreover, we also became aware of some related works (all unpublished) of Mikis's, and Postnikovs'.They also proposed a similar construction of $RLL = LLR^*$ with the dynamical R-matrix independently.

Appendix A. The normal order relation for basic operators

The normal order relations for the screening currents $E(v)$ and $F(v)$ of q-deformed Virasoro algebra are

$$E(v_1)E(v_2) = x^{\frac{4(r-1)v_1}{r}} t(v_2 - v_1) : E(v_1)E(v_2) :$$

$$F(v_1)F(v_2) = x^{\frac{4rv_1}{r-1}} t'(v_2 - v_1) : F(v_1)F(v_2) :$$

$$E(v_1)F(v_2) = \frac{x^{-4v_1}}{(1 - xx^{2(v_2-v_1)})(1 - x^{-1}x^{2(v_2-v_1)})} : E(v_1)F(v_2) :$$

$$F(v_2)E(v_1) = \frac{x^{-4v_2}}{(1 - xx^{2(v_1-v_2)})(1 - x^{-1}x^{2(v_1-v_2)})} : E(v_1)F(v_2) : .$$

The normal order relations for the basic operators in type I and type II vertex operators read as

$$\eta_1(v_1)\eta_1(v_2) = x^{\frac{(r-1)v_1}{r}} g_1(v_2 - v_1) : \eta_1(v_1)\eta_1(v_2) :$$
$$\eta_1(v_1)\xi(v_2) = x^{\frac{2(1-r)v_1}{r}} s(v_2 - v_1) : \eta_1(v_1)\xi(v_2) :$$
$$\xi(v_2)\eta_1(v_1) = x^{\frac{2(1-r)v_2}{r}} s(v_1 - v_2) : \eta_1(v_1)\xi(v_2) :$$
$$\xi(v_1)\xi(v_2) = x^{\frac{4(r-1)v_1}{r}} t(v_2 - v_1) : \xi(v_1)\xi(v_2) :$$
$$\eta_1'(v_1)\eta_1'(v_2) = x^{\frac{rv_1}{r-1}} g_1'(v_2 - v_1) : \eta_1'(v_1)\eta_1'(v_2) :$$
$$\eta_1'(v_1)\xi'(v_2) = x^{\frac{-2rv_1}{r-1}} s'(v_2 - v_1) : \eta_1'(v_1)\xi'(v_2) :$$
$$\xi'(v_2)\eta_1'(v_1) = x^{\frac{-2rv_2}{r-1}} s'(v_1 - v_2) : \eta_1'(v_1)\xi'(v_2) :$$
$$\xi'(v_1)\xi'(v_2) = x^{\frac{4rv_1}{r-1}} t'(v_2 - v_1) : \xi'(v_1)\xi'(v_2) :$$
$$\eta_1(v_1)\xi'(v_2) = (x^{2v_1} - x^{2v_2}) : \eta_1(v_1)\xi'(v_2) :$$
$$\xi'(v_2)\eta_1(v_1) = (x^{2v_2} - x^{2v_1}) : \eta_1(v_1)\xi'(v_2) :$$
$$\eta_1'(v_1)\xi(v_2) = (x^{2v_1} - x^{2v_2}) : \eta_1'(v_1)\xi(v_2) :$$
$$\xi(v_2)\eta_1'(v_1) = (x^{2v_2} - x^{2v_1}) : \eta_1'(v_1)\xi(v_2) :$$
$$\xi(v_1)\xi'(v_2) = \frac{x^{-4v_1}}{(1 - xx^{2(v_2-v_1)})(1 - x^{-1}x^{2(v_2-v_1)})} : \xi(v_1)\xi'(v_2) :$$
$$\xi'(v_2)\xi(v_1) = \frac{x^{-4v_2}}{(1 - xx^{2(v_1-v_2)})(1 - x^{-1}x^{2(v_1-v_2)})} : \xi(v_1)\xi'(v_2) :$$

where

$$g_1(v) = \frac{\{x^{2+2v}\}\{x^{2+2r+2v}\}}{\{x^{4+2v}\}\{x^{2r+2v}\}} \qquad s(v) = \frac{(x^{2r-1+2v}; x^{2r})}{(x^{1+2v}; x^{2r})}$$
$$t(v) = (1 - x^{2v})\frac{(x^{2+2v}; x^{2r})}{(x^{2r-2+2v}; x^{2r})}$$
$$g_1'(v) = \frac{\{x^{2v}\}'\{x^{2+2r+2v}\}'}{\{x^{2r+2v}\}'\{x^{2+2v}\}'} \qquad \{z\}' = (z; x^{2(r-1)}, x^4)$$
$$s'(v) = \frac{(x^{2r-1+2v}; x^{2(r-1)})}{(x^{-1+2v}; x^{2(r-1)})} \qquad t'(v) = (1 - x^{2v})\frac{(x^{-2+2v}; x^{2(r-1)})}{(x^{2r+2v}; x^{2(r-1)})}.$$

Appendix B. The proof of the commutation relations between $K_i^{\pm}(v)$, $E(v)$ and $F(v)$

The proof is a direct substitution of the Gauss decomposition of $L^{\pm}$-operators in the relations (24)–(26). (One should be careful to deal with the order between the dynamical R-matrices and the Guass components of $L^{\pm}$-operators.) Here, we give the proof of equation (34) as an example. After some straightforward calculations, one can obtain the following commutation relations between the partial currents $E^{\pm}(v)$ and $F^{\pm}(v)$

$$[E^{\pm}(v_2), F^{\pm}(v_1)] = K_1^{\pm}(v_1)^{-1}\frac{c^{\pm}(v_1 - v_2)}{b^{\pm}(v_1 - v_2)}K_2^{\pm}(v_1) - K_2^{\pm}(v_2)\frac{c'^{\pm}(v_1 - v_2)}{b'^{\pm}(v_1 - v_2)}K_1^{\pm}(v_2)^{-1}$$
$$[E^{-}(v_2), F^{+}(v_1)] = K_1^{+}(v_1)^{-1}\frac{c^{+}(v_1 - v_2 + \frac{c}{2})}{b^{+}(v_1 - v_2 + \frac{c}{2})}K_2^{+}(v_1)$$
$$- K_2^{-}(v_2)\frac{c'^{+}(v_1 - v_2 - \frac{c}{2})}{b'^{+}(v_1 - v_2 - \frac{c}{2})}K_1^{-}(v_2)^{-1}$$

$$[E^+(v_2), F^-(v_1)] = K_1^-(v_1)^{-1}\frac{c^-(v_1-v_2-\frac{c}{2})}{b^-(v_1-v_2-\frac{c}{2})}K_2^-(v_1)$$
$$-K_2^+(v_2)\frac{c'^-(v_1-v_2+\frac{c}{2})}{b'^-(v_1-v_2+\frac{c}{2})}K_1^+(v_2)^{-1}.$$

Then, we have

$$[E(v_1), F(v_2)] = K_2^-\left(v_1+\frac{c}{2}\right)\left\{\frac{c'^+(v_1-v_2+\frac{c}{2})}{b'^+(v_1-v_2+\frac{c}{2})}-\frac{c'^-(v_1-v_2+\frac{c}{2})}{b'^-(v_1-v_2+\frac{c}{2})}\right\}K_1^-\left(v_1+\frac{c}{2}\right)^{-1}$$
$$+K_2^+(v_1)\left\{\frac{c'^-(v_1-v_2-\frac{c}{2})}{b'^-(v_1-v_2-\frac{c}{2})}-\frac{c'^+(v_1-v_2-\frac{c}{2})}{b'^+(v_1-v_2-\frac{c}{2})}\right\}K_1^+(v_1)^{-1}.$$

Using the following identity

$$\frac{c'^+(v+\mathrm{i}\epsilon)}{b'^+(v+\mathrm{i}\epsilon)}-\frac{c'^-(v-\mathrm{i}\epsilon)}{b'^-(v-\mathrm{i}\epsilon)}=\frac{1}{x-x^{-1}}\delta(v)\frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi}-1]_{r-c}}$$

when

$$\epsilon\longrightarrow 0\theta'_t=(x-x^{-1})\frac{\partial}{\partial v}[v]_t|_{v=0}$$

we find equation (34).

Appendix C. The proof of the commutation relations (37)–(43)

First we prove the relations (37) and (38). From the properties (27) and (28), using the Gauss decomposition for $L^\pm$-operators we have

$$R^\pm(v_1,\hat{\pi})E(v_2)=E(v_2)R^\pm(v_1,\hat{\pi}-2c)\qquad R^\pm(v_1,\hat{\pi})F(v_2)=F(v_2)R^\pm(v_1,\hat{\pi})$$
$$R^{*\pm}(v_1,\hat{\pi})E(v_2)=E(v_2)R^{*\pm}(v_1,\hat{\pi})\qquad R^{*\pm}(v_1,\hat{\pi})F(v_2)=F(v_2)R^{*\pm}(v_1,\hat{\pi}-2c).$$

Note that with the relations (33a) and (33b), we have

$$E(v_1)E(v_2)=\frac{[v_1-v_2-1]_r}{[v_1-v_2+1]_r}E(v_2)E(v_1)$$
$$F(v_1)F(v_2)=\frac{[v_1-v_2+1]_{r-c}}{[v_1-v_2-1]_{r-c}}F(v_2)F(v_1).$$

In order to obtain the relations between $H^\pm(v)$, the relations between $H^\pm(v)$ and $E(v)$, $F(v)$, we should deal with equation (30b). It can be rewritten as

$$\frac{a^\pm(v_1-v_2)}{a'^\pm(v_1-v_2)}K_2^\pm(v_2)\frac{[v_1-v_2+1]_{r-c}[\hat{\pi}]_{r-c}}{[v_1-v_2]_{r-c}[\hat{\pi}-1]_{r-c}}K_1^\pm(v_1)^{-1}$$
$$=K_1^\pm(v_1)^{-1}\frac{[v_1-v_2+1]_r[\hat{\pi}]_r}{[v_1-v_2]_r[\hat{\pi}-1]_r}K_2^\pm(v_2).$$

Taking the limit of $v_1\longrightarrow v_2$ in both sides of the above equation, we have

$$K_2^\pm(v)\frac{[1]_{r-c}[\hat{\pi}]_{r-c}}{\theta'_{r-c}[\hat{\pi}-1]_{r-c}}K_1^\pm(v)^{-1}=K_1^\pm(v)^{-1}\frac{[1]_r[\hat{\pi}]_r}{\theta'_r[\hat{\pi}-1]_r}K_2^\pm(v).$$

Then we have two equivalent definitions of $H^\pm(v)$

$$H^+(v)=K_2^-\left(v+\frac{c}{4}\right)\frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta'_{r-c}[\hat{\pi}-1]_{r-c}}K_1^-\left(v+\frac{c}{4}\right)^{-1}$$

$$= K_1^-\left(v+\frac{c}{4}\right)^{-1}\frac{[\hat{\pi}]_r[1]_r}{\theta_r'[\hat{\pi}-1]_r}K_2^-\left(v+\frac{c}{4}\right) \tag{78}$$

$$H^-(v) = K_2^+\left(v+\frac{c}{4}\right)\frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta_{r-c}'[\hat{\pi}-1]_{r-c}}K_1^+\left(v+\frac{c}{4}\right)^{-1}$$

$$= K_1^+\left(v+\frac{c}{4}\right)^{-1}\frac{[\hat{\pi}]_r[1]_r}{\theta_r'[\hat{\pi}-1]_r}K_2^+\left(v+\frac{c}{4}\right). \tag{79}$$

From these two equivalent definitions of $H^{\pm}(v)$, one can check equations (40)–(41). Here, we give the proof of equation (42) as an example

$$\begin{aligned}
H^+(v_1)H^+(v_2) = K_2^-&\left(v_1+\frac{c}{4}\right)\frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta_{r-c}'[\hat{\pi}-1]_{r-c}} \\
&\times K_1^-\left(v_1+\frac{c}{4}\right)^{-1}K_1^-\left(v_2+\frac{c}{4}\right)^{-1}\frac{[\hat{\pi}]_r[1]_r}{\theta_r'[\hat{\pi}-1]_r}K_2^-\left(v_2+\frac{c}{4}\right) \\
&= K_2^-\left(v_1+\frac{c}{4}\right)\frac{[\hat{\pi}]_{r-c}[1]_{r-c}}{\theta_{r-c}'[\hat{\pi}-1]_{r-c}}\frac{a'^-(v_1-v_2)}{a^-(v_1-v_2)}K_1^-\left(v_2+\frac{c}{4}\right)^{-1} \\
&\times K_1^-\left(v_1+\frac{c}{4}\right)^{-1}\frac{[\hat{\pi}]_r[1]_r}{\theta_r'[\hat{\pi}-1]_r}K_2^-\left(v_2+\frac{c}{4}\right) \\
&= \frac{[v_2-v_1]_{r-c}[1]_{r-c}}{\theta_{r-c}'[v_2-v_1+1]_{r-c}}a^-(v_2-v_1)K_2^-\left(v_1+\frac{c}{4}\right)\frac{1}{b'^-(v_2-v_1)}K_1^-\left(v_2+\frac{c}{4}\right)^{-1} \\
&\times K_1^-\left(v_1+\frac{c}{4}\right)^{-1}\frac{[\hat{\pi}]_r[1]_r}{\theta_r'[\hat{\pi}-1]_r}K_2^-\left(v_2+\frac{c}{4}\right) \\
&= \frac{[v_2-v_1]_{r-c}[v_1-v_2]_r[1]_{r-c}[1]_r}{\theta_{r-c}'\theta_r'[v_2-v_1+1]_{r-c}[v_1-v_2+1]_r} \\
&\times K_1^-\left(v_2+\frac{c}{4}\right)^{-1}\frac{a^-(v_2-v_1)}{b^-(v_2-v_1)}K_2^-\left(v_2+\frac{c}{4}\right) \\
&\times K_2^-\left(v_1+\frac{c}{4}\right)\frac{a'^-(v_1-v_2)}{b'^-(v_1-v_2)}K_1^-\left(v_1+\frac{c}{4}\right)^{-1} \\
&= \frac{[v_1-v_2-1]_r[v_1-v_2+1]_{r-c}}{[v_1-v_2+1]_r[v_1-v_2-1]_{r-c}}H^+(v_2)H^+(v_1).
\end{aligned}$$

Here we have used the identity

$$\frac{a'^{\pm}(v_1-v_2)}{a^{\pm}(v_1-v_2)} = \frac{a^{\pm}(v_2-v_1)}{a'^{\pm}(v_2-v_1)}.$$

Similarily, we can prove the other relations among $H^{\pm}(v)$. The following identites are very useful for the proof

$$a^+(v)a^-(-v) = 1 \qquad a'^+(v)a'^-(-v) = 1.$$

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